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POLYHEDRAL PROOF METHODS IN COMBINATORIAL OPTIMIZATION

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Introduction

Often a combinatorial optimization problem can be formulated as an integer linear programming problem:

(1) $$\max cx, \quad \text{subject to } Ax \le b, \quad x \text{ integral.}$$

In some cases, the *LP-relaxation*, obtained from (1) by deleting the integrality condition on x, has already an integral optimum solution – without requiring so a priori. This allows to apply purely linear programming methods to solve the combinatorial optimization problem.

This is a basis of the polyhedral methods in combinatorial optimization. As an example consider the following *optimum branching problem*. Suppose we are given n locations $1, \dots, n$, together with distances d_{ij} between them (not-necessarily symmetric, i.e., $d_{ij} \ne d_{ji}$ is allowed). We wish to choose certain of the connections ij in such a way that they together form a rooted directed spanning tree, with root 1 (a 1-*branching*), and so that the sum of the distances of the connections chosen is as small as possible. In terms of integer linear programming:

(2) $$\min \sum_{i,j=1}^{n} d_{ij}x_{ij},$$

$$\begin{aligned} \text{subject to } & \sum_{i=1}^{n} x_{ij} = 1 && (j=2,\dots,n), \\ & \sum_{i\notin C, j\in C} x_{ij} \ge 1 && (\emptyset \ne C \subseteq \{2,\dots,n\}), \\ & x_{ij} \ge 0 && (i,j=1,\dots,n), \\ & x_{ij} \text{ integer} && (i,j=1,\dots,n). \end{aligned}$$

It was shown by Edmonds [1967] that in this program the integrality condition can be skipped without changing the minimum value. It means that we can solve the optimum branching problem by solving the linear programming problem

$$\text{(3)} \qquad \min \sum_{i,j=1}^{n} d_{ij}x_{ij},$$

$$\text{subject to } \sum_{i=1}^{n} x_{ij}=1 \qquad (j=2,\dots,n),$$

$$\sum_{i\notin C, j\in C} x_{ij}\geq 1 \qquad (\emptyset\neq C\subseteq\{2,\dots,n\}),$$

$$x_{ij}\geq 0 \qquad (i,j=1,\dots,n).$$

Note that this LP-problem has exponentially many constraints, so that a too straightforward application of LP-techniques will be not efficient. Yet, one can show the polynomial-time solvability of the problem with the ellipsoid method. This method gives that if for any given x the constraints $Ax\leq b$ can be checked in polynomial time, then also the linear program $\max\{cx \mid Ax\leq b\}$ can be solved in polynomial time (cf. Grötschel, Lovász and Schrijver [1981]). Here *checking* means: given x, decide if it satisfies $Ax\leq b$, and if not, find a violated constraint. This can be done, sometimes, faster than by testing each of the inequalities one by one.

E.g., the constraints in (3) can be checked as follows. Given $(x_{ij} \mid i,j=1,\dots,n)$, first check the first and third class of constraints in (3), altogether n^2+n-1 constraints. If these conditions are fulfilled, check the remaining constraints by considering x_{ij} as a 'capacity' function on the arcs ij, and by determining, for each $j\neq 1$, a cut C_j separating 1 from j of minimum capacity (with Dinits' version of the Ford–Fulkerson max-flow min-cut algorithm). If each of the cuts C_j has capacity at least 1, then all constraints in (3) are fulfilled. Otherwise, we have a cut C_j of capacity less than 1, yielding a violated inequality in (3).

Note that this algorithm checks the constraints in (3) in time polynomially bounded by n and the size of x, while the constraint system itself has size exponential in n.

The ellipsoid method now gives that also the minimum (3) can be determined in time polynomially bounded by n and the size of $(d_{ij} \mid i,j=1,\dots,n)$. Therefore, the optimum branching problem is also polynomially solvable. (In fact, Edmonds [1967] also gave a direct polynomial algorithm.)

This gives one motivation for studying polyhedral methods. The ellipsoid method proves polynomial solvability, it is however not (yet) a practical method. The polyhedral methods can be used to deduce also practical methods from the LP-representation of the combinatorial problem, e.g., by imitating the simplex method or by a primal-dual approach (see Papadimitriou and Steiglitz [1982]).

A theoretical corollary of many polyhedral results is the 'facial' description of combinatorial polyhedra. E.g., Edmonds' theorem mentioned above is equivalent to the statement that the feasible region of (3) has integral vertices. Equivalently, that the feasible region of (3) is the convex hull of the characteristic vectors of the 1-branchings.

A second theoretical interpretation is in terms of a *min–max relation*. Edmonds' theorem says that (2) and (3) are equal for each choice of the d_{ij}. By the duality theorem of linear programming, for $d_{ij} \geq 0$, (3) is equal to

$$\text{(4)} \qquad \max \sum_{C} y_C,$$

$$\text{subject to } \sum_{C,\, i \notin C,\, j \in C} y_C \leq d_{ij} \qquad (i, j = 1, \ldots, n),$$

$$y_C \geq 0 \qquad (\emptyset \neq C \subseteq \{2, \ldots, n\}).$$

Therefore, Edmonds' theorem is equivalent to: the minimum value in the optimum branching problem is equal to the maximum (4). It was shown moreover by Fulkerson [1974] that, if the d_{ij} are integer, then the maximum (4) also has an integral optimum solution.

So polyhedral methods can yield polynomial-time solvability, practical algorithms and theoretical facts. For *NP-complete* problems the situation is a little different, although polyhedral methods can be helpful.

First observe that *each* integer linear programming problem can be viewed as an LP-problem, since the convex hull of the integral vectors in a convex polyhedron is itself a convex polyhedron. However, the inequalities necessary for describing this last polyhedron can be very complicated: it was shown by Karp and Papadimitriou [1980] that if a class of ILP-problems is NP-complete, and if we assume NP $\neq$ co-NP (as is generally believed), then among the inequalities necessary for the corresponding LP-problem there are those for which a proof of validity requires exponential time. That is, the convex hull of the integral solutions has facets which cannot be shown even to be valid in polynomial time. So if NP $\neq$ co-NP, there is no hope for a nice ILP-formulation of any NP-complete problem where the integrality conditions are superfluous. All necessary inequalities can be found in principle, viz. by the cutting plane procedure of Gomory [1958], but this is not a polynomial-time method.

As an example, consider the NP-complete *traveling salesman problem*:

$$\text{(5)} \qquad \min \sum_{i,j=1}^{n} d_{ij} x_{ij},$$

$$\text{subject to } \sum_{i=1}^{n} x_{ij} = 1 \qquad (j = 1, \ldots, n),$$

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad (i = 1, \ldots, n),$$

$$\sum_{i \notin C,\, j \in C} x_{ij} \geq 1 \qquad (\emptyset \neq C \subseteq \{2, \ldots, n\}),$$

$$x_{ij} \geq 0 \qquad (i, j = 1, \ldots, n),$$

$$x_{ij} \text{ integer} \qquad (i, j = 1, \ldots, n).$$

Removing the integrality condition generally will change the minimum, and finding all inequalities necessary to be added to make the integrality condition superfluous seems infeasible. We can use however the LP-relaxation of (5):

$$\text{(6)} \qquad \begin{aligned} &\min \sum_{i,j=1}^{n} d_{ij}x_{ij}, \\ &\text{subject to } \sum_{i=1}^{n} x_{ij} = 1 && (j=1,\dots,n), \\ &\sum_{j=1}^{n} x_{ij} = 1 && (i=1,\dots,n), \\ &\sum_{i\notin C, j\in C} x_{ij} \geq 1 && (\emptyset \neq C \subseteq \{2,\dots,n\}), \\ &x_{ij} \geq 0 && (i,j=1,\dots,n). \end{aligned}$$

This minimum can be used as a lower bound in a branch-and-bound method for the traveling salesman problem. Again one can show, with a method similar to the one used for (3), that (6) is solvable in time polynomially bounded by the size of d_{ij} and by n, but this is with the ellipsoid method, and not practical. Among the practical methods proposed to solve (6) is the *Lagrange-approach* of Held and Karp [1962]: The *Lagrange function* is:

$$\text{(7)} \qquad \begin{aligned} F(\lambda) := \min &\sum_{i,j=1}^{n} d_{ij}x_{ij} + \sum_{i=1}^{n} \lambda_i\left(1 - \sum_{j=1}^{n} x_{ij}\right), \\ &\text{subject to } \sum_{i=1}^{n} x_{ij} = 1 && (j=1,\dots,n), \\ &\sum_{i\notin C, j\in C} x_{ij} \geq 1 && (\emptyset \neq C \subseteq \{2,\dots,n\}), \\ &x_{ij} \geq 0 && (i,j=1,\dots,n), \end{aligned}$$

for $\lambda \in \mathbb{R}^n$. Note that (7) is a linear program of type (3), so that we can add integrality conditions on x_{ij} without changing the minimum value. Moreover, for each fixed λ, $F(\lambda)$ can be calculated in polynomial time. F is a concave function, whose maximum is exactly equal to the minimum value of (6), which can be seen by writing down the dual programs of the programs (6) and (7). Held and Karp applied the so-called *subgradient method* for maximizing F.

An alternative method for solving (6) is to first solve (6) with the simplex method while deleting the third set of constraints. Next we check (e.g. with the max-flow min-cut algorithm), whether the optimal solution satisfies the third set of constraints. If so we are finished. If not, add a violated constraint to the linear program, and do some dual pivot steps to obtain a new LP-optimum. Next check if this new solution satisfies the third set of constraints. If so we are finished. If not, repeat as before. This algorithm terminates, or can be stopped before termination if no significant progress is made anymore – in that case we can use the current LP-optimum value as a lower bound in the branch-and-bound method for the traveling salesman method. This *cutting plane approach* was used successfully by Crowder, Grötschel and Padberg for problems with up to 318 cities.

Having given some introduction and motivation to polyhedral methods, we now discuss some of the proof methods.

1. Elementary polyhedral methods

Elementary, though nontrivial properties of polyhedra can be very helpful in polyhedral combinatorics.

A set P of vectors in $\mathbb{R}^n$ is called a *polyhedron* if

$$P = \{x \mid Ax \leq b\} \tag{8}$$

for some system $Ax \leq b$ of linear inequalities. Here and in the sequel, by using notation like $Ax \leq b$, we shall assume implicitly compatibility of sizes, so that if A is an $m \times n$-matrix, then b is a column vector of m components.

A set P of vectors is called a *polytope* if it is the convex hull of finitely many vectors. Fundamental is the following intuitively clear, but nontrivial to prove, result, which is essentially due to Farkas [1894], Minkowski [1896] and Weyl [1935]:

(9) P is a polytope iff P is a bounded polyhedron.

An element x^* of P is a *vertex* if it is not a convex combination of other elements of P. Each vertex of $P = \{x \mid Ax \leq b\}$ is determined by setting n linearly independent constraints in $Ax \leq b$ to equality.

Application 1. *Perfect matchings in bipartite graphs and doubly stochastic matrices.* A square matrix $A = (a_{ij})$ of order n is called *doubly stochastic* if

(10)
$$\sum_{i=1}^{n} a_{ij} = 1 \quad (j = 1, \ldots, n),$$
$$\sum_{j=1}^{n} a_{ij} = 1 \quad (i = 1, \ldots, n),$$
$$a_{ij} \geq 0 \quad (i, j = 1, \ldots, n).$$

A *permutation matrix* is a $\{0, 1\}$-matrix with in each row and in each column exactly one 1. Birkhoff [1947] and Von Neumann [1953] showed:

(11) A is doubly stochastic iff A is a convex combination of permutation matrices.

Proof. By induction on the order n of A, the case $n = 1$ being trivial. Consider the polytope P, in n^2 dimensions, of all doubly stochastic matrices. So P is defined by (10). To prove (11), it suffices to show that each vertex of P is a convex combination of permutation matrices. So let A be a vertex of P. Then n^2 linearly independent constraints among (10) are satisfied by A with equality. As the first $2n$ constraints in (10) are linearly dependent, it follows that at least n^2-2n+1 of the a_{ij} are 0. So A has a row with $n-1$ 0's and one 1. Without loss of generality, $a_{11}=1$. So all other entries in the first row and in the first column are 0. Then deleting the first row and first column of A gives a doubly stochastic matrix of order $n-1$, which by our induction hypothesis is a convex combination of permutation matrices of order $n-1$. Therefore, A itself is a convex combination of permutation matrices of order n. □

It follows that (10) has integral vertices. Hence in any integer linear program over (10) we can delete the integrality conditions. Therefore, the *optimal assignment problem*

$$\text{(12)}\qquad \begin{aligned} &\min \sum_{i,j=1}^{n} c_{ij}x_{ij},\\ &\text{subject to } \sum_{i=1}^{n} x_{ij}=1 \qquad (j=1,\dots,n),\\ &\qquad\qquad \sum_{j=1}^{n} x_{ij}=1 \qquad (i=1,\dots,n),\\ &\qquad\qquad x_{ij}=0 \text{ or } 1 \qquad (i,j=1,\dots,n), \end{aligned}$$

is just a linear program.

Another corollary is that each regular bipartite graph G of degree $r\geq 1$ has a perfect matching (Frobenius [1912], Kőnig [1916]). To see this, let $\{1,\dots,n\}$ and $\{n+1,\dots,2n\}$ be the colour classes of G, and define the $n\times n$-matrix $A=(a_{ij})$ by:

$$\text{(13)}\qquad a_{ij}:=\frac{1}{r}\cdot(\text{number of edges connecting } i \text{ and } n+j).$$

Then A is doubly stochastic, and hence, by (11), there exists a permutation matrix $B=(b_{ij})$ such that $a_{ij}>0$ if $b_{ij}=1$. Therefore, the edges connecting i and $n+j$ if $b_{ij}=1$ form a perfect matching in G. Deleting these edges and repeating this argument gives that the edges of G can be split into perfect matchings.

Application 2. *The perfect matching polytope.* Let $G=(V,E)$ be an undirected graph, with $|V|$ even, and let P be the associated *perfect matching polytope*, i.e., P is the convex hull of the characteristic vectors (in $\{0,1\}^E$) of the perfect matchings in G. *Edmonds' matching polyhedron theorem* [1965] states that P is the polytope defined by

$$\text{(14)}\qquad \begin{aligned} &x_e\geq 0 && (e\in E),\\ &x(\delta(v))=1 && (v\in V),\\ &x(\delta(W))\geq 1 && (W\subseteq V,\ |W| \text{ odd}). \end{aligned}$$

Here $\delta(W)$ is the set of edges of G intersecting W in exactly one point, $\delta(v) := \delta(\{v\})$, and $x(F) := \sum_{e \in F} x_e$ whenever $F \subseteq E$.

Let Q be the set of vectors in $\mathbb{R}^E$ satisfying (14). As the characteristic vector of any perfect matching satisfies (14), it follows that $P \subseteq Q$ – the content of Edmonds' theorem is the converse inclusion; equivalently, that the polytope defined by (14) has integral vertices only.

Edmonds' matching polyhedron theorem. *The perfect matching polytope is determined by the inequalities* (14).

Proof. Let G be a smallest graph with $Q \not\subseteq P$ (that is, with $|V| + |E|$ as small as possible). Let x be a vertex of Q not contained in P. Then $0 < x_e < 1$ for all e in E – otherwise we could delete e from G to obtain a smaller counterexample. Moreover, $|E| > |V|$ – otherwise, either G is disconnected (in which case one of the components of G will be a smaller counterexample), or G has a point v of degree 1 (in which case the edge e incident with v has $x_e = 1$), or G is an even circuit (for which the theorem is trivial).

Since x is a vertex, there are $|E|$ linearly independent constraints among (14) satisfied by x with equality. Hence there exists a $W \subseteq V$ with $|W|$ odd, $|W| \geq 3$, $|V \setminus W| \geq 3$, and $x(\delta(W)) = 1$. Let G_1 and G_2 arise from G by contracting W and $V \setminus W$, respectively. Let x_1 and x_2 be the corresponding projections of x onto the edge sets of G_1 and G_2, respectively. Since x_1 and x_2 satisfy the inequalities (14) for the smaller graphs G_1 and G_2, it follows that x_1 and x_2 can be decomposed as convex combinations of perfect matchings in G_1 and G_2, respectively. These decompositions can be easily glued together to form a decomposition of x as a convex decomposition of perfect matchings in G, contradicting our assumption.

[This glueing can be done, e.g., as follows. By the rationality of x (as it is a vertex of Q), there exists a natural number K such that, for $i = 1, 2$, Kx_i is the sum of the incidence vectors of the perfect matchings $F_1^i, \ldots, F_K^i$ of G_i (possibly with repetitions). Since for each e in $\delta(W)$, e is contained in Kx_e of the F_j^1 as well as in Kx_e of the F_j^2, we may assume that, for each $j = 1, \ldots, K$, F_j^1 and F_j^2 intersect in an edge of $\delta(W)$. So $F_j^1 \cup F_j^2$ is a perfect matching in G, and Kx is the sum of the incidence vectors of these perfect matchings. Hence x itself is a convex combination of perfect matchings in G.] $\square$

Finding a minimum weighted perfect matching in G is clearly the same as solving

(15) $$\min \sum_{e \in E} w_e x_e$$

$$\begin{aligned} \text{subject to } x_e &\geq 0 && (e \in E), \\ x(\delta(v)) &= 1 && (v \in V), \\ x(\delta(W)) &\geq 1 && (W \subseteq V,\ |W| \text{ odd}), \\ x_e &\text{ integer} && (e \in E). \end{aligned}$$

By Edmonds' theorem, we can delete the integrality condition, and just solve the LP-relaxation. Padberg and Rao [1980] showed that the constraints in (15) can be checked in polynomial time – hence, with the ellipsoid method, the minimum can be calculated in polynomial time. Edmonds [1965] gave a direct polynomial-time algorithm, the famous blossom-algorithm, which in fact yields the matching polyhedron theorem as a by-product.

2. LP-duality and complementary slackness

Consider the following equations:

(16) $$\max\{cx \mid Ax \leq b\} = \min\{yb \mid y \geq 0,\ yA = c\}.$$

and

(17) $$\max\{cx \mid x \geq 0,\ Ax \leq b\} = \min\{yb \mid y \geq 0,\ yA \geq c\}.$$

The Duality Theorem of linear programming states that (16) (similarly (17)) holds provided that at least one of the two optima exists.

Moreover, there are the *complementary slackness* conditions: if x and y satisfy $Ax \leq b$ and $y \geq 0$, $yA = c$, then

(18) x and y are optimal in (16) iff
for each j: $y_j > 0$ implies $(Ax)_j = b_j$.

Similarly, if x and y satisfy $x \geq 0$, $Ax \leq b$ and $y \geq 0$, $yA \geq c$ then:

(19) x and y are optimal in (17) iff
for each j: $y_j > 0$ implies $(Ax)_j = b_j$, and
for each i: $x_i > 0$ implies $(yA)_i = c_i$.

Application 3. *Max-flow min-cut.* Let $D = (V, A)$ be a directed graph, let $r, s \in V$, and $c: A \to \mathbb{R}_+$. Then the Duality Theorem of linear programming yields:

(20)
$$\begin{array}{llll}
\max x(\delta^+(r)) - x(\delta^-(r)) & & = \min \sum_{a \in A} c_a y_a & \\
\text{subject to} & & \text{subject to} & \\
x(\delta^+(v)) = x(\delta^-(v)) & (v \in V,\ v \neq r, s) & y_a \geq 0 & (a \in A), \\
0 \leq x_a \leq c_a & (a \in A) & z_v \in \mathbb{R} & (v \in V), \\
 & & -z_v + z_w + y_a \geq 0 & (a = (v, w) \in A), \\
 & & z_r = 1,\ z_s = 0. &
\end{array}$$

Here $\delta^+(v)$ and $\delta^-(v)$ denote the sets of arcs leaving v, and entering v, respectively. The maximum (20) can be seen as the maximum amount of r-s-flow subject to the capacity constraint c.

Now let x, y, z be a, possibly fractional, optimal solution for the optima in (20). Define

(21) $$W := \{v \in V \mid z_v \geq 1\}.$$

Then $r \in W$, $s \notin W$. Let $\delta^+(W)$ and $\delta^-(W)$ denote the sets of arcs leaving W and entering W, respectively. If $a=(v,w) \in \delta^+(W)$, then $y_a \geq z_v - z_w > 0$, and hence, by complementary slackness, $x_a = c_a$. If $a=(v,w) \in \delta^-(W)$, then $y_a + z_w - z_v \geq z_w - z_v > 0$, and hence, again by complementary slackness, $x_a = 0$. Hence:

(22) $$x(\delta^+(r)) - x(\delta^-(r)) = \sum_{v \in W} (x(\delta^+(v)) - x(\delta^-(v))) = x(\delta^+(W)) - x(\delta^-(W)) = c(\delta^+(W)).$$

So the amount of flow is equal to the capacity of the cut $\delta^+(W)$. That is, we have the famous max-flow min-cut theorem of Ford and Fulkerson [1956] and Elias, Feinstein and Shannon [1956]:

(23) The maximum amount of r-s-flow subject to capacity c, is equal to the minimum capacity of an r-s-cut.

By replacing y, z by $\tilde{y}, \tilde{z}$ with

(24)
$$\begin{aligned} \tilde{y}_a &= 1 \quad \text{if } a \in \delta^+(W), \\ \tilde{y}_a &= 0 \quad \text{otherwise}, \\ \tilde{z}_v &= 1 \quad \text{if } v \in W, \\ \tilde{z}_v &= 0 \quad \text{otherwise}, \end{aligned}$$

we obtain an integral optimum solution for the minimum in (20). If c is integral, also the maximum in (20) has an integral optimum solution, which is a result of Dantzig [1951] – see Application 6 below.

Application 4. *Edge-colourings.* Let $G=(V,E)$ be an undirected bipartite graph, and consider the LP-duality equation:

(25)
$$\begin{aligned} &\max \sum_{e \in E} x_e && = \min \sum_{M} y_M, \\ &\text{subject to} && \text{subject to} \\ &\sum_{e \in M} x_e \leq 1 \quad (M \text{ matching}) && \sum_{M \ni e} y_M \geq 1 \quad (e \in E), \\ &x_e \geq 0 \quad (e \in E) && y_M \geq 0 \quad (M \text{ matching}). \end{aligned}$$

Suppose we know that the maximum here always has an integral optimum solution (in Application 11 we shall see that this indeed holds). We show that this implies that also the minimum has an integral optimum solution. Let y be any, possibly fractional, optimum solution for the minimum. Let N be a matching with $y_N > 0$.

By complementary slackness, any optimum solution x for the maximum (25) has $\sum_{e\in N} x_e = 1$. Therefore, $\mu_H = \mu_G - 1$, where μ_G denotes the common value of (25), and μ_H the common value of (25) with respect to the graph H obtained from G by deleting the edges in N. Now by induction, for H the minimum (25) has an integral optimum solution y. Adding $y_M = 1$ gives an integral optimum solution for the minimum (25) with respect to G.

Note that the fact that both optima (25) have integral optimum solutions, is equivalent to the Frobenius-Kőnig theorem:

(26) The maximum degree in a bipartite graph G is equal to the minimum number of colours needed to colour the edges of G so that no two edges of the same colour meet in a vertex.

3. Total unimodularity

A matrix is called *totally unimodular* if each of its subdeterminants is 0, +1 or −1. In particular, each entry is 0, +1 or −1. The link of total unimodularity with combinatorial optimization was laid by Hoffman and Kruskal [1956] who showed that if A is totally unimodular and b is an integral vector, then $\max\{cx \mid Ax \leq b\}$ has an integral optimum solution, for each vector c for which the optimum exists. Equivalently, the polyhedron $\{x \mid Ax \leq b\}$ is integral. This is not difficult to see: Any nonsingular submatrix of A has integral inverse, and therefore any system of linear equations derived from $Ax \leq b$ has an integral solution.

In fact, Hoffman and Kruskal showed: A is totally unimodular iff A is integral and the polyhedron $\{x \geq 0 \mid Ax \leq b\}$ has integral vertices only, for each integral vector b.

There is the following characterization of total unimodularity due to Ghouila-Houri [1962]:

(27) A is totally unimodular iff each subcollection R of the rows of A can be split into two classes R_1 and R_2 such that the sum of the rows in R_1, minus the sum of the rows in R_2, is a vector with entries 0, +1 and −1 only.

A famous characterization of total unimodularity was given by Seymour [1980], yielding a polynomial-time algorithm for testing total unimodularity.

Application 5. *Optimal assignment.* Let A be the incidence matrix of a bipartite graph, i.e., A is a $\{0,1\}$-matrix, whose rows can be split into two classes R_1 and R_2 so that each column has exactly one 1 in R_1 and exactly one 1 in R_2. It is not difficult to see that A is totally unimodular. A consequence is what we showed in Application 1: In (12) the integrality conditions can be deleted. Another consequence is that the following equality holds between two ILP-optima, for any bipartite graph $G = (V, E)$:

(28)
$$\max \sum_{e \in E} x_e = \min \sum_{v \in V} y_v$$

subject to $\quad$ subject to

$$\sum_{e \ni v} x_e \leq 1 \quad (v \in V) \qquad \sum_{v \in e} y_v \geq 1 \quad (e \in E),$$
$$x_e \geq 0 \quad (e \in E) \qquad y_v \geq 0 \quad (v \in V),$$
$$x_e \text{ integer} \qquad y_v \text{ integer}.$$

This follows from the fact that by total unimodularity of the constraint matrix, we may delete the integrality conditions, and that the two LP-optima are equal by the LP-Duality Theorem.

Note that an optimum solution for the maximum in (28) is the characteristic vector of a matching in G, and that an optimum solution of the minimum in (28) is the characteristic vector of an edge-covering point set. Therefore, (28) is equivalent to the well-known Kőnig-Egerváry Theorem:

(29) The maximum size of a matching in a bipartite is equal to the minimum size of an edge-covering point set.

Similarly, weighted versions follow.

Application 6. *Network flows.* Let A be the incidence matrix of a directed graph. Then A is totally unimodular. A little more general: any $\{0, \pm 1\}$-matrix with at most one $+1$ and at most one -1 in each column is totally unimodular.

This implies that the minimum (20) has an integral optimum solution, which fact we also proved as Application 3. Now we know moreover, if c is integral, also the maximum has an integral optimum solution. This fact was first shown by Dantzig [1951]:

(30) If the capacities c are integers, there is an integral optimum flow.

Similarly, a min–max relation for minimum cost flows follows.

4. Total dual integrality

The concept of total dual integrality was motivated by Edmonds and Giles [1977] through the following result. Suppose we are given a rational system $Ax \leq b$ of linear inequalities with b integral, and consider the LP-duality equation

(31) $$\max\{cx \mid Ax \leq b\} = \min\{yb \mid y \geq 0,\ yA = c\}.$$

Suppose the minimum has an integral optimum solution y *for each integral vector* c *for which the minimum is finite.* Then the maximum also has an integral optimum solution, for each such c. This last statement is equivalent to $Ax \leq b$ defining an integral polyhedron.

A system $Ax \leq b$ is called *totally dual integral* if the minimum in (31) has an integral optimum solution y for each c as above. Edmonds and Giles' result thus says that any totally dual integral system $Ax \leq b$ with b integral, defines an integral polyhedron.

The result of Edmonds and Giles is not difficult to show in the special case where $P := \{x \mid Ax \leq b\}$ is *pointed*, i.e., where each minimal face of P is a vertex (see Hoffman [1974]). Suppose a vertex x is not integral, say x_1 is not an integer. We can find integral objective functions c and c' such that both c and c' attain their maximum over P in x, and such that $c' - c = (1, 0, \ldots, 0)$. Since for c and c' the minimum (31) has an integral optimum solution and since b is integral, in both cases the minimum value is an integer. These minimum values are cx and $c'x$, and therefore, also $cx - c'x = x_1$ is an integer, contradicting our assumption.

This also shows the following special case of total dual integrality. Let A be a rational matrix and let b be an integral vector. If the maximum in

(32) $$\min\{cx \mid x \geq 0,\ Ax \geq b\} = \max\{yb \mid y \geq 0,\ yA \leq c\}$$

has an integral optimum solution for each integral vector c for which the maximum is finite, then the same holds for the minimum.

Application 7. *Branchings and rooted cuts.* Let $D = (V, A)$ be a directed graph, and let $r \in V$. An *r-branching* is a set T of arcs of D forming a rooted directed spanning tree, with root r. That is, T contains no circuit and each vertex $s \neq r$ is entered by exactly one arc in T. A *cut rooted in r* or an *r-cut* is a set of arcs of the form $\delta^-(W)$ with $\emptyset \neq W \subseteq V \setminus r$.

It is immediate that each r-branching intersects each r-cut. Moreover, the minimal r-cuts are exactly the minimal sets intersecting all r-branchings, and vice versa.

Fulkerson [1974] (cf. Edmonds [1967]) proved the following min–max equation.

Fulkerson's optimum branching theorem. *For any 'length' function $l: A \to \mathbb{Z}_+$, the minimum length of an r-branching is equal to the maximum number t of r-cuts $C_1, \ldots, C_t$ (repetition allowed), such that no arc a is in more than $l(a)$ of the C_i.*

Before we prove the theorem, observe the following. Let B be the matrix with columns indexed by A, and with rows the characteristic vectors of the r-cuts. Then the theorem states that for any $l: A \to \mathbb{Z}_+$, the optima in

(33) $$\min\{lx \mid x \geq 0,\ Bx \geq \mathbf{1}\} = \max\{y\mathbf{1} \mid y \geq 0,\ yB \leq l\}$$

are attained by integral optimum solutions. Here $\mathbf{1}$ denotes an all-one vector. By the theory of total dual integrality, it suffices to show that the maximum in (33) has an integer optimum solution.

Proof. Let y be an optimum solution for the maximum in (33), such that

(34) $$\sum_{\emptyset \neq W \subseteq V \setminus r} y_{\delta^-(W)} \cdot |W|^2$$

is as large as possible. Such a y exists by reason of compactness. Now let

(35) $$\mathscr{F}:=\{W\subseteq V \mid y_{\delta^-(W)}>0\}.$$

Then $\mathscr{F}$ is *laminar*, i.e., if $U, W\in\mathscr{F}$ then $U\subseteq W$ or $W\subseteq U$ or $U\cap W=\emptyset$. For suppose to the contrary that $U\not\subseteq W\not\subseteq U$ and $U\cap W\neq\emptyset$. Let $\varepsilon:=\min\{y_{\delta^-(U)}, y_{\delta^-(W)}\}>0$. Let the vector y' be given by:

(36) $$y'_{\delta^-(U)}:=y_{\delta^-(U)}-\varepsilon, \qquad y'_{\delta^-(U\cap W)}:=y_{\delta^-(U\cap W)}+\varepsilon,$$
$$y'_{\delta^-(W)}:=y_{\delta^-(W)}-\varepsilon, \qquad y'_{\delta^-(U\cup W)}:=y_{\delta^-(U\cup W)}+\varepsilon,$$

and let y' coincide with y in the other coordinates. Then $y'\geq 0$, $y'B\leq yB$, and $y'\mathbf{1}=y\mathbf{1}$, so y' again is an optimum solution in (33). However, (34) is augmented, contradicting the maximality of (34).

Now let B_0 be the submatrix of B consisting of those rows of B corresponding to sets in $\mathscr{F}$. Then B_0 is totally unimodular, as can be seen with Ghouila–Houri's characterization (27), using the laminarity of $\mathscr{F}$.

Now we have

(37) $$\max\{z\mathbf{1} \mid z\geq 0, zB_0\leq l\}=\max\{y\mathbf{1} \mid y\geq 0, yB\leq l\}.$$

Indeed, $\leq$ is trivial (by extending z with 0's), while $\geq$ follows from the fact that the second maximum in (37) is attained by the vector y above, which has 0's outside B_0.

Since B_0 is totally unimodular, the first maximum, and hence also the second maximum has an integer optimum solution. □

So although the constraints $x\geq 0$, $Bx\geq\mathbf{1}$ generally are not totally unimodular, integral optimum solutions are shown by proving that in the optimum the active constraints can be chosen to be totally unimodular. This method of proof is an example of a general technique for deriving total dual integrality – see Edmonds & Giles [1977] and Hoffman & Oppenheim [1978].

Edmonds [1967] and Fulkerson [1974] designed a polynomial-time algorithm for finding a shortest branching. The polynomial solvability also follows with the ellipsoid method, as the constraints for the maximum in (33):

(38) $$x_a\geq 0 \qquad (a\in A),$$
$$x(\delta^-(W))\geq 1 \qquad (\emptyset\neq W\subseteq V\setminus r),$$

can be checked in polynomial time, although there are exponentially many r-cuts involved; for as in the Introduction above, we can consider x as a capacity function, and find a minimum capacitated cut.

Application 8. *Directed cuts and their coverings.* Let $D=(V,A)$ be a directed graph. A *directed cut* is a set of arcs of the form $\delta^-(W)$, where $\emptyset\neq W\neq V$ and $\delta^+(W)=\emptyset$. A (*directed cut*) *covering* is a set of arcs intersecting each directed cut. It follows that a set E of arcs is a covering, iff the contraction of the arcs in E makes D strongly connected.

By a method similar to that of proving Fulkerson's optimum branching theorem, one can show that the system

$$(39) \qquad \begin{aligned} &x_a \geq 0 && (a \in A), \\ &x(\delta^-(W)) \geq 1 && (\emptyset \neq W \neq V,\ \delta^+(W) = \emptyset), \end{aligned}$$

is totally dual integral. So the polyhedron determined by (39) has all its vertices integral, each being the characteristic vector of a directed cut covering.

The total dual integrality of (39) is equivalent to the following theorem of Lucchesi and Younger [1978] (cf. Lovász [1976]):

Lucchesi–Younger theorem. *The minimum size of a directed cut covering is equal to the maximum number of pairwise disjoint directed cuts.*

Since the system (39) can be checked in polynomial time, again with the help of Ford and Fulkerson's max-flow min-cut algorithm, also minimum length directed cut coverings can be found in polynomial time, with ellipsoids. Direct polynomial algorithms were given by Karzanov [1979], Lucchesi [1976] and Frank [1981].

5. Blocking polyhedra

Blocking and anti-blocking are variants of the classical polarity of polyhedra. It was shown by Fulkerson [1970, 1971, 1972] that these relations have interesting combinatorial implications.

The basis of the theory of blocking polyhedra is as follows. Let $a_1, \ldots, a_m$ and $b_1, \ldots, b_t$ be vectors in $\mathbb{R}^n_+$ such that:

$$(40) \qquad \{x \in \mathbb{R}^n_+ \mid a_1^{\mathrm{T}} x \geq 1, \ldots, a_m^{\mathrm{T}} x \geq 1\} = \mathrm{conv.hull}\{b_1, \ldots, b_t\} + \mathbb{R}^n_+ =: P.$$

Then the *blocking polyhedron* $b(P)$ of P is defined by

$$(41) \qquad b(P) := \{y \in \mathbb{R}^n_+ \mid x^{\mathrm{T}} y \geq 1 \text{ for each } x \text{ in } P\},$$

and satisfies:

$$(42) \qquad b(P) = \{y \in \mathbb{R}^n_+ \mid b_1^{\mathrm{T}} y \geq 1, \ldots, b_t^{\mathrm{T}} y \geq 1\} = \mathrm{conv.hull}\{a_1, \ldots, a_m\} + \mathbb{R}^n_+.$$

So for $b(P)$ the roles of the a_i and b_j are interchanged compared with P. So the facets (vertices, respectively) of P correspond to the vertices (facets, respectively) of $b(P)$. Moreover, $b(b(P)) = P$.

Note that (40) is equivalent to:

$$(43) \qquad \text{for each } l \in \mathbb{R}^n_+\text{:} \quad \min\{lb_1, \ldots, lb_t\} = \max\{y\mathbf{1} \mid y \geq 0,\ yA \leq l\},$$

where A is the matrix with rows $a_1^{\mathrm{T}}, \ldots, a_m^{\mathrm{T}}$. This equivalence follows by writing down the dual program for the maximum in (43). Similarly, (42) is equivalent to:

(44) for each $w \in \mathbb{R}_+^n$: $\min\{wa_1, \dots, wa_m\} = \max\{z\mathbf{1} \mid z \geq 0, zB \leq w\}$,

where B is the matrix with rows $b_1^{\mathrm{T}}, \dots, b_t^{\mathrm{T}}$. Since (40) and (42) are equivalent, also (43) and (44) are equivalent: one class of min–max relations implies another, and vice versa.

It follows from the ellipsoid method that if $a_1, \dots, a_m$ and $b_1, \dots, b_t$ are related as above, then

(45) for each $l \in \mathbb{R}_+^n$ $\min\{lb_1, \dots, lb_t\}$ can be determined in polynomial time, iff for each $w \in \mathbb{R}_+^n$ $\min\{wa_1, \dots, wa_m\}$ can be determined in polynomial time,

also if t and m are exponentially large with respect to the 'original' problem.

Application 9. *Shortest paths and network flows.* The theory of blocking polyhedra gives another proof of the max-flow min-cut theorem. Let $D = (V, A)$ be a directed graph, and let $r, s \in V$. Let $a_1, \dots, a_m$ be the characteristic vectors of the r-s-cuts; so these are vectors in $\{0, 1\}^A$. Let $b_1, \dots, b_t$ be the characteristic vectors of the r-s-paths, again in $\{0, 1\}^A$.

We first show that (43) holds. First suppose $l: A \to \mathbb{Z}_+$, and let k be the length of a shortest r-s-path. For each i with $1 \leq i \leq k$, define

(46) $V_i := \{v \in V \mid \text{there is an } r\text{-}v\text{-path of length} < i\}$.

Then $V_1 \subseteq V_2 \subseteq \dots \subseteq V_k$, and $r \in V_i$, $s \notin V_i$. Let χ_j denote the characteristic vector of the r-s-cut $\delta^+(V_j)$. Then each χ_j occurs among the a_i, and $\chi_1 + \chi_2 + \dots + \chi_k \leq l$. Therefore, (43) holds. Next, for rational-valued l, (43) follows from the integral case by multiplying l with a large enough natural number. For real-valued l, (43) follows by continuity.

So (43) holds, and hence, by the theory above, (44) also holds. But this is the max-flow min-cut theorem: zB is an r-s-flow of value $z\mathbf{1}$ subject to the capacity w, while the minimum in (44) is the minimum capacity of an r-s-cut.

Since a shortest path can be found in polynomial time (with Dijkstra's algorithm), it follows from the ellipsoid method that a minimum capacitated cut can also be determined in polynomial time – here polynomial means: polynomially bounded by the sizes of D and c, not by the *number* of paths or cuts.

Application 10. *Branchings and rooted cuts.* Let $D = (V, A)$ be a directed graph, and let $r \in V$. Let $a_1, \dots, a_m$ be the characteristic vectors of the r-cuts, and let $b_1, \dots, b_t$ be the characteristic vectors of the r-branchings (cf. Application 7).

From Application 7 we know that (43) holds. Therefore, by the theory of blocking polyhedra, also (44) holds. It says that the minimum capacity of an r-cut is equal to the maximum value of $\lambda_1 + \dots + \lambda_m$ for which there exist r-branchings $T_1, \dots, T_m$ such that for each arc a, the sum of the λ_j for which a belongs to T_j is at most c_a.

Edmonds [1973] showed that if c is integral, we can take the λ_i integral. It means that the system

(47) $\sum_{a \in T} x_a \geq 1$ (T r-branching),

$x_a \geq 0$ ($a \in A$)

is totally dual integral. In particular (in fact, equivalently), the minimum number of arcs in an r-cut is equal to the maximum number of pairwise disjoint r-branchings (disjoint in the sense of having no common arcs). The proofs however (cf. also Lovász [1976], Tarjan [1976]) are combinatorial and/or algorithmical, and not polyhedral.

6. Anti-blocking polyhedra

A theory of anti-blocking polyhedra was also developed by Fulkerson [1971, 1972]. Let $a_1, \ldots, a_m$ and $b_1, \ldots, b_t$ be vectors in $\mathbb{R}^n_+$ such that

(48) $\{x \in \mathbb{R}^n_+ \mid a_1^{\mathrm{T}} x \leq 1, \ldots, a_m^{\mathrm{T}} x \leq 1\} = (\text{conv.hull}\{b_1, \ldots, b_t) + \mathbb{R}^n_-) \cap \mathbb{R}^n_+ =: P.$

Then the *anti-blocking polyhedron* of P is defined by

(49) $a(P) := \{y \in \mathbb{R}^n_+ \mid x^{\mathrm{T}} y \leq 1 \text{ for each } x \text{ in } P\},$

and satisfies:

(50) $a(P) = \{y \in \mathbb{R}^n_+ \mid b_1^{\mathrm{T}} y \leq 1, \ldots, b_t^{\mathrm{T}} y \leq 1\}$

$= (\text{conv.hull}\{a_1, \ldots, a_m\} + \mathbb{R}^n_-) \cap \mathbb{R}^n_+.$

So again facets and vertices are interchanged, and we have another variant of the classical polarity. It follows that $a(a(P)) = P$.

Note that (48) is equivalent to:

(51) for each $l \in \mathbb{R}^n_+$: $\max\{lb_1, \ldots, lb_t\} = \min\{y\mathbf{1} \mid y \geq 0, yA \geq l\},$

where A is the matrix with rows $a_1^{\mathrm{T}}, \ldots, a_m^{\mathrm{T}}$. This equivalence follows by writing down the dual program for the minimum in (51). Similarly, (50) is equivalent to:

(52) for each $w \in \mathbb{R}^n_+$: $\max\{wa_1, \ldots, wa_m\} = \min\{z\mathbf{1} \mid z \geq 0, zB \geq w\},$

where B is the matrix with rows $b_1^{\mathrm{T}}, \ldots, b_t^{\mathrm{T}}$. Since (48) and (50) are equivalent, also (51) and (52) are equivalent: one class of min–max relations implies another, and vice versa.

Again, it follows from the ellipsoid method that if $a_1, \ldots, a_m$ and $b_1, \ldots, b_t$ are related as above, then

(53) for each $l \in \mathbb{R}^n_+$ $\min\{lb_1, \ldots, lb_t\}$ can be determined in polynomial time, iff for each $w \in \mathbb{R}^n_+$ $\min\{wa_1, \ldots, wa_m\}$ can be determined in polynomial time,

also if t and m are exponentially large with respect to the 'original' problem.

Application 11. *Stars and matchings.* In Application 5 we saw that for any bipartite graph $G=(V,E)$, the polytope defined by:

$$\text{(54)} \qquad x_e \geq 0 \qquad (e \in E),$$
$$x(\delta(v)) \leq 1 \qquad (v \in V),$$

has integral vertices. (Here $\delta(v)$ denotes the set of edges incident with v, and $x(B') := \sum_{e \in B'} x_e$.) So the vertices are exactly the characteristic vectors of matchings of G. Therefore, taking $a_1, \ldots, a_m$ to be the characteristic vectors of the 'stars' $\delta(v)$, and $b_1, \ldots, b_t$ the characteristic vectors of the matchings, we know that (48) holds. Therefore, (50) also holds, i.e.

$$\text{(55)} \qquad y_e \geq 0 \qquad (e \in E),$$
$$y(M) \leq 1 \qquad (M \text{ matching}),$$

defines a polytope whose vertices are the characteristic vectors of the stars. So the maximum in the LP-duality equation

$$\text{(56)} \qquad \max \sum_e y_e \qquad = \min \sum_M z_M$$
$$\text{subject to} \qquad \text{subject to}$$
$$y_e \geq 0 \quad (e \in E) \qquad z_M \geq 0 \quad (M \text{ matching}),$$
$$y(M) \leq 1 \quad (M \text{ matching}) \qquad \sum_{M \ni e} z_M \geq 1 \quad (e \in E),$$

has an integral optimum solution, namely the incidence vector of a star. So the maximum is equal to the maximum degree of G. In Application 4 we saw that this implies that also the minimum has an integral optimum solution: hence it is equal to the minimum number of matchings needed to cover E, i.e., it is the minimum number of colours needed to colour the edges of G such that no two edges of the same colour meet in a vertex of G. So (56) gives the Frobenius-Kőnig theorem.

Application 12. *Perfect graphs.* Perfect graphs were introduced by Berge [1961, 1962]. Consider the following numbers for an undirected graph $G=(V,E)$:

(57) $\omega(G) :=$ the clique number of $G =$ maximum size of a clique;

$\gamma(G) :=$ the colouring number of $G =$ the minimum number of colours needed to colour the vertices of G such that no two adjacent vertices have the same colour (i.e., the minimum number of cocliques needed to cover V);

$\alpha(G) :=$ the coclique number of $G =$ the maximum size of a coclique (= set of pairwise non-adjacent vertices);

$\bar{\gamma}(G) :=$ the clique covering number = the minimum number of cliques needed to cover V.

Obviously, $\omega(G)\le\gamma(G)$, $\alpha(G)\le\bar{\gamma}(G)$, $\omega(G)=\alpha(\bar{G})$, and $\bar{\gamma}(G)=\gamma(\bar{G})$, where $\bar{G}$ denotes the complementary graph of G (which has vertex set V, two vertices being adjacent in $\bar{G}$ iff they are not adjacent in G). The circuit on 5 vertices shows that the inequalities can be strict.

Now G is called *perfect* if $\omega(G')=\gamma(G')$ for each induced subgraph G' of G.

Examples of perfect graphs are: (1) Bipartite graphs (trivially); (2) Complements of bipartite graphs (by a theorem of Kőnig, which can be derived from the total unimodularity of the incidence matrix of a bipartite graph – see Application 5 above); (3) Line graphs of bipartite graphs (by the Kőnig–Egerváry theorem – see Application 5); (4) Complements of line graphs of bipartite graphs (by the Frobenius–Kőnig theorem – see Application 11); (5) Comparability graphs (which, by definition, arise from a partial order $(V,\le)$, two vertices being adjacent iff they are comparable – the perfectness is easy); (6) Complements of comparability graphs (by a theorem of Dilworth [1950]).

It was conjectured by Berge [1961, 1962] and proved by Lovász [1972] that the complement of each perfect graph is perfect again, which implies several other min-max relations. We give a polyhedral proof of this theorem, due to Fulkerson [1972], Lovász [1972] and Chvátal [1975]. To this end, define for any undirected graph $G=(V,E)$ the *clique polytope* as the convex hull of the cliques in G, i.e., of their characteristic vectors. Clearly, any vector x in the clique polytope satisfies

(58) (i) $x_v\ge 0$ $(v\in V)$,

(ii) $x(S)\le 1$ $(S\subseteq V,\ S$ coclique),

as the characteristic vector of each clique satisfies (58). The circuit on 5 vertices shows that generally (58) can be larger than the clique polytope. Chvátal [1975] showed that the clique polytope coincides with (58) if and only if G is perfect. This can be seen to imply the perfect graph theorem.

First observe the following. Let $Ax\le\mathbf{1}$ denote the inequality system (58) (ii). So the rows of A are the characteristic vectors of the cocliques. Then it follows directly from the definition of perfectness that G is perfect iff the optima in

(59) $$\max\{wx\mid x\ge 0,\ Ax\le\mathbf{1}\}=\min\{y\mathbf{1}\mid y\ge 0,\ yA\ge w\}$$

have integral optimum solutions, for each $\{0,1\}$-vector w.

Chvátal's Theorem. *G is perfect iff its clique polytope is determined by* (58).

Proof. (I) First suppose G is perfect. For $w: V\to\mathbb{Z}_+$, let c_w denote the maximum weight of a clique. To prove that the clique polytope is given by (58), it suffices to show that

(60) $$c_w=\max\{wx\mid x\ge 0,\ Ax\le\mathbf{1}\}$$

for each $w: V\to\mathbb{Z}_+$. This will be done by induction on $\sum_{v\in V}w_v$.

If w is a $\{0,1\}$-vector, then (60) follows from the note preceding the statement of Chvátal's theorem. So we may assume that $w_u \geq 2$ for some vertex u. Let $e_u = 1$ and $e_v = 0$ if $v \neq u$. Replacing w by $w-e$ in (59) and (60), gives, by induction, a vector $y \geq 0$ such that $yA \geq w-e$ and $y\mathbf{1} = c_{w-e}$. We may assume $yA = w-e$. Since $(w-e)_u \geq 1$, there is a coclique S with $y_S > 0$ and $u \in S$. Let a be the characteristic vector of S. Note that $a \leq w-e$, since $yA = w-e$.

Then $c_{w-a} < c_w$. For suppose $c_{w-a} = c_w$. Let C be any clique with $(w-a)(C) = c_{w-a}$. Since $c_{w-a} = c_w$, $a(C) = 0$. On the other hand, since $w-a \leq w-e \leq w$, we know that $(w-e)(C) = c_{w-e}$, and hence, by complementary slackness, $a(C) > 0$, a contradiction.

Therefore,

$$(61) \qquad \begin{aligned} c_w = 1 + c_{w-a} &= 1 + \max\{(w-a)x \mid x \geq 0,\ Ax \leq \mathbf{1}\} \\ &\geq \max\{wx \mid x \geq 0,\ Ax \leq \mathbf{1}\} \end{aligned}$$

implying (60).

(II) Conversely, suppose that the clique polytope is determined by (58), i.e., that the maximum in (59) is attained by a clique, for each w. To show that G is perfect it suffices to show that the minimum in (59) also has an integral optimum solution for each $\{0,1\}$-valued w. This will be done by induction on $\sum_{v \in V} w_v$.

Let w be $\{0,1\}$-valued, and let $\tilde{y}$ be a, not-necessarily integral, optimum solution for the minimum in (59). Let S be a coclique with $\tilde{y}_S > 0$, and let a be its characteristic vector (we may assume $a \leq w$). Then the common value of

$$(62) \qquad \max\{(w-a)x \mid x \geq 0,\ Ax \leq \mathbf{1}\} = \min\{y\mathbf{1} \mid y \geq 0,\ yA \geq (w-a)\}$$

is less than the common value of (59), since by complementary slackness, each optimum solution x in (59) has $ax = 1$. However, the values in (59) and (62) are integers (since by assumption the maxima have integral optimum solutions). Hence they differ by exactly one. Moreover, by induction the minimum in (62) has an integral optimum solution y. Increasing component y_S of y by 1, gives an integral optimum solution in (59). □

The theory of anti-blocking polyhedra now gives directly the perfect graph theorem of Lovász [1972]:

Perfect graph theorem. *The complement of a perfect graph is perfect again.*

Proof. If G is perfect, by Chvátal's theorem, the clique polytope P of G is defined by (58). Hence, by the theory of anti-blocking polyhedra, the coclique polytope of G, i.e., the clique polytope of $\bar{G}$ is defined by (58) after replacing coclique by clique, i.e., coclique of $\bar{G}$. Applying Chvátal's theorem again gives that $\bar{G}$ is perfect. □

So for any undirected graph, $\omega(G') = \gamma(G')$ for each induced subgraph, iff $\alpha(G') = \bar{\gamma}(G')$ for each induced subgraph.

It is conjectured by Berge [1969] that a graph is perfect iff it has no induced subgraph isomorphic to an odd circuit of length at least five or its complement. This *strong perfect graph conjecture* is still unproved.

It was shown in Grötschel, Lovász and Schrijver [1981] that a clique of maximum size and a minimum vertex colouring in a perfect graph can be found in polynomial time.

7. Cutting planes

For any polyhedron P, define P_I as the convex hull of the integral vectors in P. It is not difficult to show (and trivial if P is bounded) that P_I is a polyhedron again. Generally it is a difficult problem to find the inequalities defining P_I. Karp and Papadimitriou [1982] showed that generally P_I has some 'difficult' facets, at least if NP$\neq$co-NP.

The cutting plane method, developed by Gomory [1958], is a non-polynomial method to find the facets of P_I – see Chvátal [1973] and Schrijver [1980].

Obviously, if H is the affine half-space $\{x \mid cx \le d\}$, where c is a nonzero integral vector whose components are relatively prime integers, then

(63) $$H_I = \{x \mid cx \le \lfloor d \rfloor\}.$$

Geometrically, H_I arises from H by shifting the bounding hyperplane until it contains integral vectors. Now define for any polyhedron P:

(64) $$P' := \bigcap_{H \supseteq P} H_I,$$

where the intersection ranges over all affine half-spaces H as above with $H \supseteq P$. As $P \subseteq H$ implies $P_I \subseteq H_I$, it follows that $P_I \subseteq P'$. So

(65) $$P \supseteq P' \supseteq P'' \supseteq P''' \supseteq \cdots \supseteq P_I.$$

It can be shown that if P is a rational polyhedron (i.e., defined by rational inequalities), then P' is a rational polyhedron again, and that $P^{(t)} = P_I$ for some natural number t. (Here $P^{(t)}$ is the $(t+1)$-th polyhedron in (65).)

This is the theory of Gomory's famous *cutting plane method*. The successive half-spaces H_I (more strictly, their bounding hyperplanes) are called *cutting planes*.

It can be shown that, for any fixed t, the class of integer linear programs for which $P_I = P^{(t)}$ has a 'good characterization', i.e., is in NP$\cap$co-NP.

Application 13. *The matching polytope.* Let $G = (V, E)$ be an undirected graph, and let Q be the *matching polytope* of G, i.e., Q is the convex hull of the characteristic vectors of matchings in G (so $Q \subseteq \mathbb{R}^E$). Let P be the polytope defined by

(66) $$x_e \ge 0 \qquad (e \in E),$$
$$x(\delta(v)) \le 1 \qquad (v \in V).$$

Since the integral vectors in P are exactly the characteristic vectors of matchings in G, we know $Q=P_\mathrm{I}$. It is not difficult to see that P' is the polytope determined by

$$\text{(67)}\qquad \begin{aligned} &x_e \geq 0 && (e\in E),\\ &x(\delta(v))\leq 1 && (v\in V),\\ &x(\langle U\rangle)\leq \tfrac{1}{2}(|U|-1) && (U\subseteq V,\ |U| \text{ odd}).\end{aligned}$$

Edmonds [1965] showed that in fact $P'=P_\mathrm{I}=Q$. That is, (67) determines the matching polytope. One can derive this in an elementary way from the characterization of the perfect matching polytope given in Application 2.

Application 14. *The coclique polytope.* Let $G=(V,E)$ be an undirected graph, and let Q be the *coclique polytope* of G, i.e., Q is the convex hull of the characteristic vectors of cocliques (so $Q\subseteq \mathbb{R}^V$). It seems to be a difficult problem to find a set of linear inequalities determining Q. If NP $\neq$ co-NP, then Q will have 'difficult' facets.

As an approximation of the coclique polytope, let P be defined by

$$\text{(68)}\qquad \begin{aligned} &x_v\geq 0 && (v\in V),\\ &x(C)\leq 1 && (C\subseteq V,\ C \text{ clique}).\end{aligned}$$

So P is the anti-blocking polyhedron of the clique polytope – cf. Application 12. As the integral solutions for (68) are exactly the characteristic vectors of cocliques, we know $Q=P_\mathrm{I}$. Now we can ask: given G, for which t is $P^{(t)}=P_\mathrm{I}=Q$?

There is no natural number t such that $P^{(t)}=Q$ for each graph G, as was shown by Chvátal [1973]. In Application 12 we saw that the class of graphs with $P=Q$ is exactly the class of perfect graphs. Chvátal [1973] raised the question whether there exists, for each fixed t, a polynomial-time algorithm for finding the maximum size of a coclique in G, for graphs G with $P^{(t)}=Q$. This is true for $t=0$ – see Grötschel, Lovász and Schrijver [1981].

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(*) means suggested further reading.